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NONINTEGRABILITY AND THE BREAKDOWN OF HAMILTON'S PRINCIPLE

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ABSTRACT :

As long-cherished postulate of theoretical physics, Hamilton's principle (HP) defines the basis of classical mechanics and field theory. We argue here that HP is overturned in physical settings where sensitivity to initial conditions cannot be ignored. We find that the approach to chaos of nearly-integrable Hamiltonian systems sheds new light on several foundational aspects of effective field theory.

Key words: Hamilton's principle, decoherence, Hamiltonian chaos, KAM theorem, multifractals, action quantization, effective field theory.

1. INTRODUCTION

As it is known, integrable systems form the backbone of classical and quantum field theory. A Hamiltonian (conservative) system with N degrees of freedom is integrable if it has N independent commuting invariants of motion. An important attribute of this class of systems is that all interactions can be eliminated by appropriate canonical transformations. Integrability implies the existence of periodic or quasi-periodic tori in phase-space, a property that can be extended to dissipative systems [1-3, 12, 16].

Nature shows, however, that most interacting Hamiltonian systems are nonintegrable and their long-term evolution chaotic. The primary mechanism explaining the onset of Hamiltonian chaos is the Kolmogorov-Arnold-Moser (KAM) theorem, which is the perturbation theory of quasi-periodic tori applied to nearly-integrable Hamiltonian systems.

In the context of this work, we take nonintegrability to arise either from sensitivity to initial conditions or undamped perturbations outside equilibrium. While sensitivity to initial conditions describes transition to chaos via positive Lyapunov exponents, undamped perturbations generate chaos via the progressive collapse of quasi-periodic tori, fragmentation of phase-space and the emergence of fractal spacetime [4-5].

As conjectured in several publications, the mechanism of decoherence - the loss of phase information and the entropy surge in open systems – comes into play beyond the Standard Model scale and favors the transition from quantum to classical behavior. A reasonable expectation is that deep Terascale physics falls outside thermodynamic equilibrium and, in doing so, it replicates the attributes of Hamiltonian chaos [6-9].

This brief analysis points out that sensitivity to initial conditions is bound to overturn HP and, on account of decoherence, able to bridge the gap between Hamiltonian chaos and the foundations of effective field theory.

The paper is formatted as follows: next section contains a brief introduction to HP in classical field theory, with emphasis on electrodynamics and General Relativity. The breakdown of SAP due to sensitive dependence and its consequences for foundational physics are analyzed in the next couple of sections.

2. STATIONARY ACTION IN CLASSICAL FIELD THEORY

Classical field theory develops from the Lagrangian

$$L = L(\varphi, \partial_\mu \varphi, x^\mu) \tag{1}$$

and the first order variation of the action functional given by [10-11]

$$\delta S = \int_R \delta\varphi\, d^4x \{\frac{\partial L}{\partial \varphi} - \partial_\mu [\frac{\partial L}{\partial(\partial_\mu \varphi)}]\} + \int_{\partial R} d\sigma_\mu \{\frac{\partial L}{\partial(\partial_\mu \varphi)} \delta\varphi + L\delta x^\mu\} \tag{2}$$

Here, R is the four-dimensional integration domain whose boundary is ∂R. The canonical treatment of (2) posits that both field and coordinate variations vanish on ∂R , i.e.

$$\delta\varphi = \delta x^\mu = 0 \;\; \text{on } \partial R \tag{3}$$

which supplies the field equation in the standard form

$$\frac{\delta S[\varphi]}{\delta \varphi} = \Lambda(\varphi, \partial_\mu \varphi, x^\mu) = \frac{\partial L}{\partial \varphi} - \frac{\partial}{\partial x^\mu}[\frac{\partial L}{\partial(\partial_\mu \varphi)}] = 0 \tag{4}$$

Applying (2) to classical electrodynamics (ϕ → A) and General Relativity, respectively, (ϕ → g), yields

$$\int d^4x\, \delta A_\mu (J^\mu - \frac{\partial F^{\mu\nu}}{\partial x^\nu}) = 0 \tag{5}$$

$$\int d^4x \sqrt{-g}\, \delta g^{\mu\nu} [\frac{1}{2} T_{\mu\nu} - \frac{1}{16\pi G}(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R)] = 0 \tag{6}$$

(5) and (6) lead to Maxwell and Einstein equations under the textbook setting that the field variations δ Aμ and g μν δ are arbitrary and the functional differential equation (4) is satisfied [10-11, 13].

3. DEPENDENT ENDPOINT CONDITIONS

Consider the generic case where fields and spacetime coordinates vary simultaneously on R , while the boundary term goes to zero at infinity. If the first term of (2) is sampled at fixed time intervals δ t , a convenient approximation of δ S over a discrete set of sampling points i N =1,2,..., can be written as

$$\delta S = \int_R d^4x \, d\varphi \, \Lambda \propto \delta t [\sum_i \delta^i x_i \delta\varphi_i \Lambda_i] \tag{7}$$

Where

$$\delta x_1 << 1, \delta\varphi_1 << 1; \; N >> 1 \tag{8a}$$

$$\varphi_i = \varphi(x_i) \tag{8b}$$

$$\Lambda_i = \Lambda(\varphi_i, \partial^x \varphi_i, x_i) \tag{8c}$$

Sensitivity to initial conditions in coordinate and field spaces, respectively, causes exponential separation of adjacent trajectories as in

$$\delta\varphi_{i+1}(x_{i+1}) \propto \delta\varphi_i(x_i) \exp[\lambda(\varphi_{i+1} - \varphi_i)]; \quad \lambda > 0; \; i > 1 \tag{9}$$

And

$$\delta x_{i+1} \propto \delta x_i \exp[\sigma(x_{i+1} - x_i)]; \quad \sigma > 0; \; i > 1 \tag{10}$$

Coordinates and fields can only be measured to finite precision. This is to say that, in fact, there are infinitely many adjacent trajectories defined through

$$x_1 = x_1 + |X| \tag{11a}$$

$$\varphi_1 = \varphi_1 + |\Omega| \tag{11b}$$

with uncertainties upper bounded by their resolution limits respectively, that is,

$$|X| \le R_x \tag{12a}$$

$$|\Omega| \le R_\varphi \tag{12b}$$

Conditions (11) and (12) imply that all adjacent trajectories starting from points located within Rx and/or Rϕ are initially indistinguishable from each other, even though they split apart later on. It follows that the endpoint variations of both field and coordinates are no longer independent and likely to become ill-defined for sufficiently large separations (i 1 i x x + >> and ϕ ϕ i+1 >> i). Stated differently, dependent endpoint conditions induce memory-like effects and are asymptotically unpredictable. Another way to look at these observations is to acknowledge that deterministic dynamics of classical field theory no longer stands, in manifest contrast with the foundation of Maxwell's electrodynamics and General Relativity.

Although somewhat unexpected, these results are nevertheless hardly surprising. They merely confirm the long-held belief that field theories complying with (3) are effective field approximations, endowed with limited ranges of validity. They also hint that realistic modeling efforts can no longer disregard the issue of sensitive dependence or the onset of nonintegrability above the Standard Model scale [6-9, 14].

4. IMPACT ON FOUNDATIONAL PHYSICS

The line of reasoning previously outlined leads one to suspect that the breakdown of HP due to sensitive dependence must impact the foundation of effective field theories. We now briefly elaborate on three examples showing that this is indeed the case.

4.1 Action quantization

Sensitive dependence is the hallmark of the transition from deterministic behavior to chaos. Taken in this context, the approach to chaos of nearlyintegrable and multidimensional Hamiltonian systems can be shown to produce action quantization in the long-term limit [7].

4.2 Curved spacetime

Consider next the propagation of a light ray in empty space. Its trajectory follows Fermat's principle, according to which the optical path length between two endpoints is stationary with respect to variations of the path. Fermat's principle in flat spacetime is an analogue of HP and it takes the form

$$\delta S = \delta \int_a^b ds = 0 \tag{13}$$

By (7)-(12), rectilinear propagation described by (13) is no longer valid if there is sensitive dependence and the endpoints cannot be arbitrarily chosen. In this case, the most straightforward deviation from (13) can be presented as

$$\delta \int_a^b ds = \delta(s_b - s_a) \neq 0 \tag{14}$$

Consider now the propagation of the same light ray in a static gravitational field whose only non-vanishing metric potential is the temporal component g_{00} [10].

$$\delta \int_a^b \frac{ds}{\sqrt{g_{00}}} = 0 \tag{15}$$

Moreover, let the gravitational potential be created by a point source of mass M at the radial distance r , such that (c =1)

$$g_{00} = 1 + 2\varphi = 1 - 2\frac{G_N M}{r} \tag{16}$$

Inserting (16) in (15), power expanding the potential and retaining only the leading terms gives a correction to rectilinear propagation (13) having the generic form

$$\delta(s_b - s_a) = f(\varphi) = f(M, r) \neq 0 \tag{17}$$

A glance at (14) and (17) indicates that the effect of sensitive dependence applied to the propagation of light rays in flat spacetime is indistinguishable from the effect of curving light rays in gravitational fields.

It follows from this analysis that gravitation may be understood as implicit outcome of sensitive dependence and the unavoidable onset of nonintegrability, as first revealed by Poincaré's 3-body problem. This conclusion backs up the line of arguments developed in [15-17].

4.3 Four spacetime dimensions

The fragmented structure of Hamiltonian flows in phase space includes islands of stability sandwiched between ergodic regions. There are strong intermittencies associated with this regime and their characterization requires the language of multifractals (MF) and fractional dynamics [4-5]. Applying the MF geometry to the geodesic equation recovers the four dimensionality of classical spacetime [18].

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